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Chebyshev-Type Quadrature on Multidimensional Domains

J. KOREVAAR AND J. L. H. MEYERS

Faculty of Mathematics and Computer Science, Plantage Muidergracht 24, 1018 TV Amsterdam, the Netherlands

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Quadrature formulas with equal coefficients for interval and circle are combined to obtain Chebyshev-type quadrature formulas (relative to ordinary area or volume measure) for "product domains." Upper bounds for the minimal number $N = N(p)$ of nodes required for polynomial exactness to degree p readily follow. Lower bounds are obtained by projecting onto certain subsets of lower dimension and other means. The precise order of $N(p)$ is determined for square, cube, cylindrical surface, disc, and cylinder, while upper and lower bounds for the order are found for sphere and ball. Improving recent results of Bajnok and Rabau, the authors describe so-called spherical t-designs (Chebyshev-type quadrature formulas of degree t for the sphere with distinct nodes) consisting of $\mathcal{O}(t^3)$ points.

1. INTRODUCTION AND RESULTS

Let E be a compact set in $\mathbf{R}^d$ and let σ be a finite positive measure on E. A Chebyshev-type quadrature formula for E and σ is a numerical integration formula which gives equal weight to the (not necessarily distinct) nodes which we call $\xi_1, \ldots, \xi_N$,

$$\frac{1}{\sigma(E)} \int_E f(x)\, d\sigma(x) \approx \frac{1}{N} \sum_{j=1}^{N} f(\xi_j), \qquad \xi_j \in E. \tag{1.1}$$

In other words, integrals are approximated by arithmetic means of function values.

We will say that a system of nodes $\xi_1, \ldots, \xi_N$ generates a *quadrature formula* (1.1) *of degree* (at least) p if the formula is exact for all polynomials $f(x_1, \ldots, x_d)$ of total degree $\le p$. The simplest Chebyshev quadrature

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formula is the Gauss formula for the interval $[-1,1]$ with weight function $1/\sqrt{1-x^2}$. Here the m-point formula has degree $2m-1$,

$$\frac{1}{\pi}\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}} \approx \frac{1}{m}\sum_{j=1}^{m} f(\xi_j),$$

where $\xi_j = \cos((2j-1)\pi/2m)$ runs over the zeros of the Chebyshev polynomial $T_m(x) = \cos(m \arccos x)$.

In this paper we deal with simple two-dimensional surfaces E and ordinary surface measure σ. Our central question is as follows. *What is the minimal number $N = N_E(p)$ for which there is a Chebyshev-type quadrature formula* (1.1) *of degree p?* Our results concern surfaces E which can be considered as Cartesian products of intervals and/or circles in a certain parametrization. Thinking of large p we will say that $N_E(p)$ is of order p^λ,

$$N_E(p) \asymp p^\lambda$$

if there are positive constants c_1 and c_2 such that

$$c_1 p^\lambda \le N_E(p) \le c_2 p^\lambda, \qquad p \ge 1.$$

Using this terminology, the work of S. N. Bernstein [2, 3, 4] shows that for the interval $I = [-1,1]$ and $d\sigma(x) = dx$,

$$N_I(p) \asymp p^2,$$

cf. Section 2. For the unit circle $C = C(0,1)$ in $\mathbf{R}^2 \simeq \mathbf{C}$ with $d\sigma = ds$, one will immediately think of nodes at the $(p+1)$th roots of unity to obtain

$$N_C(p) = p + 1,$$

cf. Section 3. Combining these results we prove the following estimates, always using ordinary surface area:

(i) for the square $Q = I^2$ in $\mathbf{R}^2$, $N_Q(p) \asymp p^4$;

(ii) for the cylindrical surface $\mathrm{CS} = C \times I$ in $\mathbf{R}^3$, $N_{\mathrm{CS}}(p) \asymp p^3$;

(iii) for the unit disc D in $\mathbf{R}^2$, $N_D(p) \asymp p^3$;

(iv) for the unit sphere $S = S(0,1)$ in $\mathbf{R}^3$, $c_1 p^2 \le N_S(p) \le c_2 p^3$.

We derive explicit constants in all cases and exhibit Chebyshev-type quadrature formulas. The latter and the corresponding upper bounds follow directly from the product structures. Lower bounds may be obtained by projection onto a subset of lower dimension or by ad hoc means: in each case we have presented the method that gave the best result.

Numerical results of A. B. J. Kuijlaars [17] for the interval I suggest that our lower bounds in (i)–(iii) are somewhat closer than the upper bounds.

For the sphere S we do not know the precise order of $N_S(p)$, but we conjecture that $N_S(p) \asymp p^2$. This conjecture is based on a certain equivalence between Chebyshev-type quadrature on S and a "spherical Faraday cage problem" involving equal point charges, see [15]. Specifically, a Chebyshev-type quadrature formula for S with nodes $\xi_1, \dots, \xi_N$ has degree p if and only if point charges $1/N$ at these N nodes result in an electrostatic field which vanishes to order p at the origin. For the point charges problem it appears plausible that one can achieve $p \approx c\sqrt{N}$.

In combinatorics, J. J. Seidel *et al.* have introduced so-called spherical t-designs, that is, configurations of N *distinct* points $\xi_1, \dots, \xi_N$ on S for which formula (1.1) with $E = S$ has degree (at least) t, cf. [6, 12, 23]. We exhibit spherical t-designs consisting of $\mathcal{O}(t^3)$ points, improving previous results by B. Bajnok [1] and P. Rabau and Bajnok [22].

The present paper includes some comments on other sets E, such as ellipse, cube, solid cylinder, ball, and higher dimensional spheres. For the interval $[-1, 1]$ we also discuss ultraspherical measures. For the case of the torus T in $\mathbf{R}^3$, Kuijlaars has just shown that $N_T(p) \asymp p^2$, see [18].

A preliminary form of some of our results has appeared in the second author's Ph.D. thesis [20].

2. The Interval $[-1, 1]$

The best known quadrature formula is the classical Gauss formula. Let $P_m(x)$ denote the Legendre polynomial of degree $m \geq 1$ with zeros

$$\alpha_k = \alpha_k(m): \qquad 1 > \alpha_1 > \alpha_2 > \cdots > \alpha_m > -1.$$

We may write the m-point Gauss formula in the form

$$\frac{1}{2}\int_{-1}^{1} f(x)\, dx \approx \sum_{k=1}^{m} \lambda_k f(\alpha_k), \tag{2.1}$$

where the Cotes–Christoffel numbers $\lambda_k = \lambda_k(m)$ are given by

$$\lambda_k = \frac{1}{(1-\alpha_k^2)P_m'(\alpha_k)^2}.$$

Formula (2.1) has degree $2m - 1$: it is exact for all polynomials of degree $\leq 2m - 1$.

It is appropriate to observe that

$$P_m\left(\cos\frac{z}{m}\right) \to J_0(z) \qquad \text{as } m \to \infty,$$

uniformly on every compact set in the z-plane. Here J_0 is the Bessel function of order 0. Also, in terms of the first positive zero $j_1 = j_{0,1}$ of J_0,

$$m^2(1 - \alpha_1(m)) \nearrow \tfrac{1}{2}j_1^2 \approx 2.8916, \qquad m^2\lambda_1(m) \nearrow J_0'(j_1)^{-2} \approx 3.7104, \tag{2.2}$$

cf. [25] and for the monotonicity, see [2–4]; a related inequality for $m^2\lambda_1(m)$ occurs in [8].

In the thirties, Bernstein [14] obtained fundamental results on Chebyshev-type quadrature for the interval $[-1, 1]$ with constant weight function. Taking (2.2) into account, his results may be formulated as follows.

THEOREM 2.1 (Bernstein). (i) *Suppose that the nodes $x_1, \ldots, x_N$ on $[-1, 1]$ are such that the quadrature formula*

$$\frac{1}{2}\int_{-1}^{1} f(x)\,dx \approx \frac{1}{N}\sum_{j=1}^{N} f(x_j) \tag{2.3}$$

has degree (at least) $2m - 1$ $(m \geq 1)$. Then at least one of the nodes x_j is $\geq \alpha_1(m)$, and

$$N \geq \frac{1}{\lambda_1(m)} > J_0'(j_1)^2 m^2 > .269m^2.$$

(ii) *Let $m \geq 1$ and let $N_1 = N_1(2m - 1)$ be the smallest even integer $> 4\sqrt{2}(m + 1)(m + 4)$. Then there exist points $t_i \in (-1, 1)$, $t_1 > t_2 > \cdots > t_{2m-1}$, $t_{2m-i} = -t_i$ and positive integers $\mu_i = \mu_{2m-i}$ such that*

$$\frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{N_1(2m - 1)}\sum_{i=1}^{2m-1} \mu_i f(t_i) \tag{2.4}$$

for all polynomials $f(t)$ of degree $\leq 2m - 1$. Observe that

$$N_1(2m - 1) < 4\sqrt{2}(m + 1)(m + 4) + 2. \tag{2.5}$$

Formula (2.4) is a symmetric Chebyshev-type formula having $N_1(2m - 1)$ nodes in which the node t_i appears with multiplicity μ_i, $\Sigma\mu_i = N_1(2m - 1)$. In this way Bernstein proved that the smallest Cotes–Christoffel number

$\lambda_1(m)$ determines the order of the minimal number of nodes in a Chebyshev-type quadrature formula of degree $2m-1$; in the notation of Section 1,

$$N_I(2m-1) \asymp \frac{1}{\lambda_1(m)} \asymp m^2.$$

It has recently been proved by Kuijlaars [16] that the multiple nodes in (2.4) can always be split into simple nodes without losing polynomial exactness to degree $2m-1$.

For his proof of part (i) Bernstein employed two special polynomials which will also be useful to us,

$$F_1(x) = \frac{P_m(x)^2}{(x-\alpha_1)^2}, \qquad F_2(x) = (x-\alpha_1)F_1(x), \qquad \alpha_1 = \alpha_1(m). \tag{2.6}$$

Observe that $F_1(x)$ is strictly increasing for $x \geq \alpha_2$ and that by the Gauss formula,

$$\frac{1}{2}\int_{-1}^{1} F_1(x)\,dx = \lambda_1 F_1(\alpha_1) = \lambda_1 P_m'(\alpha_1)^2, \qquad \frac{1}{2}\int_{-1}^{1} F_2(x)\,dx = 0. \tag{2.7}$$

For some of our multidimensional problems it is important to know that the first part of Bernstein's Theorem extends to arbitrary positive measures σ on $[-1,1]$, see W. Gautschi [9, 10]. We will describe the situation for the normalized ultraspherical measure

$$d\sigma_\nu(x) \stackrel{\text{def}}{=} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}(1-x^2)^{\nu-1/2}\,dx, \qquad \nu > -\frac{1}{2}$$

of total mass 1. The role of the Legendre polynomials is now taken over by the ultraspherical polynomials $P_m^{(\nu)}(x)$ which we standardize as in Szegö's book [25, Sect. 4.7]. The Christoffel numbers in the corresponding Gauss formula may be derived from [25, Sect. 15.3]. Using the duplication formula for the Gamma function, the first Christoffel number for the normalized measure becomes

$$\lambda_1^{(\nu)}(m) = \frac{2\nu}{\Gamma(2\nu)}\frac{\Gamma(m+2\nu)}{\Gamma(m+1)}(1-\alpha_1^2)^{-1}P_m^{(\nu)\prime}(\alpha_1)^{-2} \qquad (\nu \neq 0),$$

where $\alpha_1 = \alpha_1^{(\nu)}(m)$ is the largest zero of $P_m^{(\nu)}(x)$.

We discuss the precise behavior of $\lambda_1^{(\nu)}(m)$ for $m \to \infty$. One may obtain an asymptotic formula for $P_m^{(\nu)}$ in terms of the Bessel function J_β with $\beta = \nu - \frac{1}{2}$, cf. [25, Sects. 8.1 and 4.7]. Also using the relation $\Gamma(m+b)/\Gamma(m) \sim m^b$ one finds that

$$m^{1-2\nu} P_m^{(\nu)}\left(\cos\frac{z}{m}\right) \to \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)} 2^{\nu-1/2} z^{-\beta} J_\beta(z), \qquad m \to \infty \quad (2.8)$$

uniformly on every compact set in the z-plane. Hence by differentiation, cf. [25, Sect. 1.71] for the Bessel function,

$$m^{1-2\nu} P_m^{(\nu)\prime}\left(\cos\frac{z}{m}\right)\left(-\sin\frac{z}{m}\right)\left(\frac{1}{m}\right) \to \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)} 2^{\nu-1/2}\left\{-z^{-\beta} J_{\beta+1}(z)\right\}.$$

By (2.8), $\alpha_1(m) \approx \cos(j_1/m)$, $1 - \alpha_1^2 \approx \sin^2(j_1/m)$, where $j_1 = j_{\beta,1}$ denotes the smallest positive zero of J_β. A computation involving the duplication formula for the Gamma function finally shows that for $m \to \infty$,

$$\begin{aligned}\frac{\Gamma(m+2\nu+1)}{\Gamma(m)} \lambda_1^{(\nu)}(m) &\to A_\nu \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{3}{2})} j_{\nu-1/2,1}^{2\nu-1} J_{\nu+1/2}\left(j_{\nu-1/2,1}\right)^{-2}.\end{aligned} \tag{2.9}$$

The left-hand side of (2.9) increases monotonically with m when $0 < \nu \le \frac{1}{2}$ (cf. [14]) and when $\nu = 1$, see (2.10) below. We can show general monotonicity for $m > m_0(\nu)$ and perhaps A_ν provides an upper bound for the left-hand side of (2.9) whenever $\nu \ge 0$; cf. [8] for $0 < \nu < 1$.

Leaving speculation aside, a general upper bound B_ν (resembling A_ν!) for the left-hand side of (2.9) may be derived from the literature. In connection with the ultraspherical Chebyshev-type quadrature problem, general upper bounds for $\lambda_1^{(\nu)}(m)$ have been investigated by L. Gatteschi [7], A. Meir and A. Sharma [19], and F. Costabile [5]. If we correct for some small errors, the final result of Gatteschi and Costabile may be stated as follows:

THEOREM 2.2. *Suppose that the nodes $x_1, \ldots, x_N$ on $[-1,1]$ are such that the quadrature formula*

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} f(x)(1-x^2)^{\nu-1/2}\,dx \approx \frac{1}{N}\sum_{j=1}^{N} f(x_j)$$

has degree (at least) $2m - 1$. *Then for* $\nu \geq 0$,

$$N \geq \frac{1}{\lambda_1^{(\nu)}(m)} \geq \frac{1}{B_\nu} \frac{\Gamma(m + 2\nu + 1)}{\Gamma(m)} \geq \frac{1}{B_\nu} m^{2\nu+1},$$

where

$$B_\nu = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{3}{2})} j_{\nu+1/2,1}^{2\nu-1} J_{\nu-1/2}(j_{\nu+1/2,1})^{-2}.$$

In Section 6 we will use the case $\nu = 1$, where the relevant polynomials are the Chebyshev polynomials of the second kind:

$$P_m^{(1)}(x) = U_m(x) = \frac{\sin(m + 1)\theta}{\sin\theta}, \qquad \cos\theta = x;$$

$$\lambda_1^{(1)}(m) = \frac{2}{m + 1} \sin^2 \frac{\pi}{m + 1}. \tag{2.10}$$

The other direction. For the ultraspherical measures $d\sigma_\nu$ with $\nu > 1/2$, Kuijlaars [16] has recently extended the second part of Bernstein's theorem, thus establishing the order estimate

$$N_{I,\sigma_\nu}(2m - 1) \asymp \frac{1}{\lambda_1^{(\nu)}(m)} \asymp m^{2\nu+1}.$$

3. The Unit Circle

Let $C = C(0, 1)$ be the unit circle in $\mathbf{R}^2 \simeq \mathbf{C}$ described by $x + iy = z = e^{i\phi}$, $-\pi < \phi \leq \pi$, with element of arc length $ds = d\phi = dz/iz$. Systems of nodes on C will be denoted by

$$(x_j, y_j) \qquad \text{or} \qquad z_j = x_j + iy_j = e^{i\phi_j}, \qquad j = 1, \ldots, N. \tag{3.1}$$

Theorem 3.1. (i) *Suppose that the nodes* (3.1) *give a Chebyshev-type quadrature formula*

$$\frac{1}{2\pi} \int_C g(x, y)\, ds \approx \frac{1}{N} \sum_{j=1}^{N} g(x_j, y_j) \tag{3.2}$$

which is exact for all polynomials $g(x, y)$ *of degree* $\leq n - 1$. *Then every closed arc of* $C(0,1)$ *of length* $2\pi/n$ *must contain a node* $z_j = e^{i\phi_j}$ *and hence* $N \geq n$.

(ii) *For $N = n$ one obtains a quadrature formula* (3.2) *which is exact to degree $n - 1$ if and only if one uses equidistant nodes on C. For all trigonometric polynomials $G(\phi)$ of order $\leq n - 1$ and every real δ,*

$$\frac{1}{2\pi}\int_C G(\phi)\,d\phi = \frac{1}{n}\sum_{j=1}^{n} G(\phi_j), \qquad \phi_j = \delta + \frac{2\pi j}{n}. \tag{3.3}$$

For the case of distinct nodes, so that one deals with "circular t-designs" where $t = n - 1$, the inequality $N \geq n$ and part (ii) are contained in the combinatorial work of P. Delsarte, J. M. Goethals, and J. J. Seidel, see [6, Theorems 5.11, 5.12 and Example 5.14]. Using complex analysis, Y. Hong [13] has obtained results on circular t-designs involving more than $t + 1 = n$ points.

Verification that equidistant nodes work. Observe that on $C(0,1)$, every polynomial $g(x, y)$ of degree $n - 1$ can be written as a trigonometric polynomial $G(\phi)$ of order $n - 1$ and conversely. It readily follows that formula (3.2) is polynomially exact to degree $n - 1$ if and only if

$$\begin{aligned}\frac{1}{2\pi}\int_C z^k \frac{dz}{iz} &\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\phi}\,d\phi \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos k\phi + i\sin k\phi)\,d\phi \\ &= \frac{1}{N}\sum_{j=1}^{N} z_j^k \equiv \frac{1}{N}\sum_{j=1}^{N} e^{ik\phi_j} \equiv \frac{1}{N}\sum_{j=1}^{N}(\cos k\phi_j + i\sin k\phi_j), \\ &\qquad k = 1, \ldots, n-1. \end{aligned} \tag{3.4}$$

The first three terms in (3.4) are equal to zero for every positive integer k. On the other hand, if one takes $N = n$ the last three terms will vanish for $1 \leq k \leq n - 1$ if one places the nodes z_j at the nth roots of unity. More generally one may rotate the system of nth roots through a fixed angle, which leads to a proof of (3.3).

For a complete proof of Theorem 3.1 and for later use we introduce the special trigonometric polynomials of order $< n$ given by

$$G_1(\phi) = G_1(\phi; n) = \frac{1 + \cos n\phi}{(\cos\phi - \cos\pi/n)^2} = \frac{1 + T_n(t)}{(t - \cos\pi/n)^2}, \tag{3.5}$$

$$G_2(\phi) = G_2(\phi; n) = G_1(\phi)(\cos\phi - \cos\pi/n), \qquad n \geq 2,$$

where $T_n(t)$ is the Chebyshev polynomial given by $T_n(\cos\phi) = \cos n\phi$. By (3.3) with $n \ge 2$, using the nodes corresponding to $\phi = \pm\pi/n$, $\pm 3\pi/n, \cdots$ on $-\pi < \phi \le \pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(\phi)\,d\phi = \frac{1}{n}\left\{G_1\left(\frac{\pi}{n}\right) + G_1\left(-\frac{\pi}{n}\right)\right\} > 0,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G_2(\phi)\,d\phi = 0. \tag{3.6}$$

Observe that $G_1(\phi) \ge 0$ for every ϕ while $G_2(\phi) > 0$ only for $\cos\phi > \cos\pi/n$.

Completion of the proof of Theorem 3.1. (a) Assuming that the hypothesis of part (i) is satisfied, we will show that every closed arc of $C(0,1)$ of length $2\pi/n$ must contain a node $z_j = e^{i\phi_j}$. Here it is sufficient to take $n \ge 2$ and to consider the arc corresponding to $-\pi/n \le \phi \le \pi/n$; the case of other arcs may be treated by "rotation" of the functions $G_i(\phi)$. Suppose then that the nodes (3.1) are such that (3.2) is exact to degree $n-1$. Then by (3.6),

$$\sum_{j=1}^{N} G_1(\phi_j) > 0, \qquad \sum_{j=1}^{N} G_2(\phi_j) = 0. \tag{3.7}$$

We may choose the values ϕ_j on $(-\pi, \pi]$ and have to show that at least one of them lies on $[-\pi/n, \pi/n]$. By (3.7) there are two possibilities: either all ϕ_j's belong to the zero set $Z(G_2)$, or $\Sigma_{j=1}^{N} G_2(\phi_j) = 0$ because there is cancellation of positive and negative values. In the first case one of the ϕ_j's must be equal to $\pm\pi/n$ or else $\Sigma_{j=1}^{n} G_1(\phi_j)$ would vanish. In the second case $G_2(\phi_j) > 0$ for at least one ϕ_j and such a point on $(-\pi, \pi]$ must belong to $(-\pi/n, \pi/n)$.

(b) Continuing under the hypothesis of part (i), it follows from (a) that $N \ge n$. Moreover, if $N = n$ the nodes must be equidistant on C or else some closed arc of length $2\pi/n$ would be free of nodes.

Remark 3.2. In the terminology of Section 1, $N_C(p) = p + 1$. One may ask what can be said about quadrature formulas of type (3.2) for curves other than circles. For the *ellipse* E in $\mathbf{R}^2$ given by $x = a\cos\phi$, $y = b\sin\phi$, $0 \le \phi < 2\pi$, results of Ya. L. Geronimus [11] suggest that $N_E(p)$ may also be of order p.

4. The Square

Let $Q = I^2$ be the closed region in $\mathbf{R}^2$ described by $-1 \le x \le 1$, $-1 \le y \le 1$. Systems of nodes in Q will be denoted by

$$(x_j, y_j), \qquad j = 1, \ldots, N. \tag{4.1}$$

Theorem 4.1. (i) *Suppose that the nodes* (4.1) *give a Chebyshev-type quadrature formula*

$$\frac{1}{4}\int_Q f(x, y)\,dx\,dy \approx \frac{1}{N}\sum_{j=1}^{N} f(x_j, y_j) \tag{4.2}$$

which is exact for all polynomials $f(x, y)$ of degree $\le p$. Then for $p \ge 1$ and $m = [(p+3)/4]$ so that $4m - 3 \le p \le 4m$ and with $\lambda_1 = \lambda_1(m)$ as in Section 2,

$$N > \frac{9}{16}\lambda_1^{-2} > \frac{9}{4^6}J_0'(j_1)^4 p^4 > .00015p^4.$$

(ii) *Let m be the least positive integer such that $2m - 1 \ge p$ and let t_i, μ_i, and $N_1(2m-1)$ provide a quadrature formula as in part* (ii) *of Bernstein's Theorem* 2.1. *Set $N(p) = N_1(2m-1)^2$. Then for all polynomials $f(x, y)$ of degree $\le p$,*

$$\frac{1}{4}\int_Q f(x, y)\,dx\,dy = \frac{1}{N(p)}\sum_{i,k=1}^{2m-1} \mu_i \mu_k f(t_i, t_k). \tag{4.3}$$

Here

$$N(p) < 2\left\{(p+4)(p+10) + \sqrt{2}\right\}^2.$$

Proof of Part (i). Suppose that formula (4.2) with the nodes (4.1) is exact to degree $p \ge 1$. For $m = [(p+3)/4]$ we define $F_1(x) = P_m(x)^2/(x - \alpha_1)^2$ and $F_2(x) = (x - \alpha_1)F_1(x)$ with $\alpha_1 = \alpha_1(m)$ as in formula (2.6). We next form two polynomials of degree $\le p$ as follows:

$$g(x, y) = F_1(x)F_1(y),$$

$$h(x, y) = (x + y - 2\alpha_1)g(x, y) = F_2(x)F_1(y) + F_1(x)F_2(y).$$

Using both (4.2) and the Gauss formula, cf. (2.7), we then find that

$$\frac{1}{N}\sum_{j=1}^{N} g(x_j, y_j) = \frac{1}{4}\int_Q g(x, y)\, dx\, dy = \lambda_1^2 g(\alpha_1, \alpha_1) > 0, \tag{4.4}$$

$$\frac{1}{N}\sum_{j=1}^{N} h(x_j, y_j) = \frac{1}{4}\int_Q h(x, y)\, dx\, dy = 0. \tag{4.5}$$

(a) We will use these results to show that at least one of the nodes (4.1) belongs to the closed triangular region

$$T = \{(x, y) \in Q\colon x + y \geq 2\alpha_1\}. \tag{4.6}$$

Indeed, in view of (4.5) there are two possibilities for the nodes (x_j, y_j): either they all lie in the zero set $Z(h)$ of h, or there are positive and negative values $h(x_j, y_j)$ which cancel each other. In the first case at least one of the nodes must belong to $Z(h) - Z(g)$, hence to the segment

$$L = \{(x, y) \in Q\colon x + y = 2\alpha_1\} \subset T,$$

or else the sum in (4.4) would vanish. In the second case $h(x_j, y_j) > 0$ for at least one node (x_j, y_j) and such a point must lie to the Northeast of L, hence in T.

(b) Renumbering the nodes we may assume that $(x_1, y_1) \in T$. Then by (4.4) and the nonnegativity of g,

$$\begin{aligned} N\lambda_1^2 g(\alpha_1, \alpha_1) &= \sum_{j=1}^{N} g(x_j, y_j) \geq g(x_1, y_1) \geq \min_T g(x, y) \\ &= g(\alpha_1, \alpha_1) \min_T \frac{F_1(x)}{F_1(\alpha_1)} \frac{F_1(y)}{F_1(\alpha_1)} \\ &= g(\alpha_1, \alpha_1)\left\{\min_T f(x) f(y)\right\}^2, \end{aligned} \tag{4.7}$$

where

$$f(x) = \frac{P_m(x)}{(x - \alpha_1) P_m'(\alpha_1)} = c_m \prod_{i=2}^{m} (x - \alpha_i), \qquad c_m > 0.$$

Note that $f(x)$ is positive, increasing and convex for $x > \alpha_2$. On T, $x \geq 2\alpha_1 - 1 > \alpha_2$ by the property $1 - \alpha_1 < \alpha_1 - \alpha_2$ of the zeros of P_m, cf. [25, Sect. 6.3]. Thus $f(x)f(y)$ is an increasing function of x on T, hence its minimum is attained on L.

We will obtain a simple lower bound for

$$\begin{aligned}\min_L f(x)f(y) &= \min_{\alpha_1 \le x \le 1} f(x)f(2\alpha_1 - x)\\ &= \min_{\alpha_1 \le x \le 1} c_m^2 \prod_{i=2}^{m} \{(x-\alpha_i)(2\alpha_1 - \alpha_i - x)\}\\ &= \min_{\alpha_1 \le x \le 1} c_m^2 \prod_{i=2}^{m} \{(\alpha_1 - \alpha_i)^2 - (x-\alpha_1)^2\}\\ &= f(1)f(2\alpha_1 - 1). \end{aligned} \tag{4.8}$$

Observe that $f(\alpha_1) = 1$ while by the differential equation for $P_m(x)$, $f'(\alpha_1) = \frac{1}{2}P_m''(\alpha_1)/P_m'(\alpha_1) = \alpha_1/(1-\alpha_1^2)$. Thus since the graph of $f(x)$ over the interval $[2\alpha_1 - 1, 1]$ lies above its tangent line at the point $(\alpha_1, f(\alpha_1))$,

$$\begin{aligned}f(1)f(2\alpha_1 - 1) &\ge \{f(\alpha_1) + f'(\alpha_1)(1-\alpha_1)\}\{f(\alpha_1) - f'(\alpha_1)(1-\alpha_1)\}\\ &= 1 - \left(\frac{\alpha_1}{1+\alpha_1}\right)^2 > \frac{3}{4}. \end{aligned} \tag{4.9}$$

Combining (4.7)–(4.9) and using (2.2) we conclude that

$$\begin{aligned}N &\ge \left(\frac{f(1)f(2\alpha_1 - 1)}{\lambda_1}\right)^2 > \frac{9}{16}\lambda_1^{-2} > \frac{9}{16}J_0'(j_1)^4 m^4\\ &\ge \frac{9}{4^6}J_0'(j_1)^4 p^4 > .00015p^4. \end{aligned}$$

Proof of Part (ii). Part (ii) will follow from the observation that under the given conditions, formula (4.3) is actually true for all monomials

$$f(x,y) = x^\alpha y^\beta \qquad \text{with } 0 \le \alpha, \beta \le 2m-1.$$

Indeed, by Bernstein's formula (2.4),

$$\begin{aligned}\frac{1}{4}\int_Q x^\alpha y^\beta \,dx\,dy &= \frac{1}{2}\int_{-1}^{1} x^\alpha \,dx \frac{1}{2}\int_{-1}^{1} y^\beta \,dy\\ &= \frac{1}{N_1(2m-1)}\sum_{i=1}^{2m-1} \mu_i t_i^\alpha \frac{1}{N_1(2m-1)}\sum_{k=1}^{2m-1}\mu_k t_k^\beta. \end{aligned}$$

The upper bound for $N(p)$ follows from inequality (2.5) for $N_1(2m-1)$, coupled with the remark that $m \le \frac{1}{2}(p+2)$.

Remark 4.2. Using the notation of Section 1 we have found that

$$N_Q(p) \asymp p^4.$$

A similar proof will show that for the cube $K = I^3$ in $\mathbf{R}^3$,

$$N_K(p) \asymp p^6.$$

In both cases, lower bounds may also be obtained by projection onto a diagonal and use of the Chebyshev–Markov–Stieltjes inequalities [25, Sect. 3.41] to estimate the relevant Christoffel number.

5. The Cylindrical Surface

Let CS be the cylindrical surface $C(0,1) \times [-1,1]$ in $\mathbf{R}^3$ described by $-1 \leq z \leq 1$, $x = \cos\phi$, $y = \sin\phi$ ($0 \leq \phi < 2\pi$ or $-\pi < \phi \leq \pi$), with surface element $d\sigma = ds\,dz = dz\,d\phi$. Systems of nodes on CS will be denoted by

$$(x_j, y_j, z_j) \simeq (z_j, \phi_j), \qquad j = 1, \ldots, N. \tag{5.1}$$

Theorem 5.1. (i) *Suppose that the nodes* (5.1) *give a Chebyshev-type quadrature formula*

$$\frac{1}{4\pi}\int_{\mathrm{CS}} f(x,y,z)\,ds\,dz \approx \frac{1}{N}\sum_{j=1}^{N} f(x_j, y_j, z_j) \tag{5.2}$$

of degree p. Then there is a constant $c > 0$ such that

$$N \geq cp^3.$$

(*For $p \geq 4$ one may take $c = .0049$, and for $p \geq 40$, $c = .01$.*)

(ii) *For $p \geq 1$, let m be the smallest integer $\geq (p+1)/2$ and choose points $t_i \in (-1,1)$, positive integers μ_i and an integer $N_1(2m-1)$ to provide a quadrature formula as in part* (ii) *of Bernstein's Theorem* 2.1. *Set*

$$z_i = t_i, \qquad \phi_k = \frac{(2k-1)\pi}{(p+1)}, \qquad N(p) = (p+1)N_1(2m-1).$$

Then for all polynomials $f(x, y, z) \simeq F(z, \phi)$ *of degree* $\leq p$,

$$\frac{1}{4\pi}\int_{CS} f(x, y, z)\, ds\, dz = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} F(z, \phi)\, dz\, d\phi$$
$$= \frac{1}{N(p)}\sum_{k=1}^{p+1}\sum_{i=1}^{2m-1} \mu_i F(z_i, \phi_k). \tag{5.3}$$

Here

$$N(p) < \sqrt{2}(p+1)\{(p+4)(p+10)+\sqrt{2}\} \leq Cp^3,$$

with $C = 1.5$ *for large* p.

Proof. (a) Let the nodes (5.1) satisfy the hypothesis of part (i), let $p \geq 1$, $q = [(p+1)/2]$. Applying formula (5.2) to $f(x, y, z) = h(z)$, where h is an arbitrary polynomial of degree $\leq 2q - 1 \leq p$, we find that

$$\frac{1}{2}\int_{-1}^{1} h(z)\, dz = \frac{1}{N}\sum_{j=1}^{N} h(z_j).$$

Hence by Bernstein's Theorem 2.1 there is a node (z_j, ϕ_j) with $z_j \geq \alpha_1(q)$, the largest zero of $P_q(t)$. We renumber the nodes so that $z_1 = \max z_j$. Rotation of the system of nodes about the z-axis will not affect the polynomial exactness of (5.2) to degree p, hence we may assume that $\phi_1 = 0$ so that

$$(x_1, y_1, z_1) \simeq (z_1, \phi_1) = (z_1, 0), \qquad z_1 \geq \alpha_1(q).$$

(b) Continuing under the hypothesis of part (i), we next take $p \geq 4$, $m \geq 1$, $n \geq 2$ and $2m + n - 2 \leq p$. Observe that $m \leq p/2 \leq q$, so that

$$z_1 \geq \alpha_1(q) \geq \alpha_1(m). \tag{5.4}$$

Letting F_1 be the polynomial of degree $2m - 2$ from (2.6) and G_1 the trigonometric polynomial of order $n - 2$ from (3.5), we define a nonnegative polynomial f on CS by

$$f(x, y, z) \simeq H(z, \phi) = F_1(z)G_1(\phi). \tag{5.5}$$

Since $\deg f < p$ we may apply formula (5.2) to obtain the inequality

$$\begin{aligned} I &= \frac{1}{4\pi}\int_{CS} f(x,y,z)\,ds\,dz = \frac{1}{4\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} H(z,\phi)\,dz\,d\phi \\ &= \frac{1}{N}\sum_{j=1}^{N} H(z_j,\phi_j) \geq \frac{1}{N}H(z_1,0) \\ &= \frac{1}{N}F_1(z_1)G_1(0) \geq \frac{1}{N}F_1(\alpha_1)\frac{2}{(1-\cos\pi/n)^2}, \qquad \alpha_1 = \alpha_1(q). \quad (5.6) \end{aligned}$$

Because $H = F_1G_1$, its integral may also be evaluated by the Gauss quadrature formula for $[-1,1]$, combined with Theorem 3.1 for the circle. The result is, cf. (2.7), (3.6),

$$\begin{aligned} I &= \frac{1}{2}\int_{-1}^{1} F_1(z)\,dz\,\frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(\phi)\,d\phi = \lambda_1(m)F_1(\alpha_1)\frac{2}{n}G_1\left(\frac{\pi}{n}\right) \\ &= \lambda_1(m)F_1(\alpha_1)\frac{n}{\sin^2\pi/n}, \qquad \alpha_1 = \alpha_1(m). \quad (5.7) \end{aligned}$$

We now combine (5.6) and (5.7). Using (5.4) and the monotonicity of $F_1(z)$, the inequality $t\cot t \geq \pi/4$ on $(0,\pi/4]$ and the second formula (2.2), one obtains the inequality

$$N \geq \frac{F_1(\alpha_1(q))}{F_1(\alpha_1(m))}\frac{8}{\pi^2 m^2\lambda_1(m)}\left(\frac{\pi}{2n}\cot\frac{\pi}{2n}\right)^2 m^2 n > \frac{1}{2}J_0'(j_1)^2 m^2 n. \quad (5.8)$$

Under the given conditions on m and n, the maximum value of m^2n is at least $p^3/27$ as can be seen by taking $m = n = [(p+2)/3] \geq p/3$. Final conclusion: for $p \geq 4$,

$$N \geq cp^3 \qquad \text{with } c = .0049. \quad (5.9)$$

For $p \geq 40$ one can use $c = .008$. Further improvement to $c = .01$ results if one exploits the fact that for our parameters, $F_1(\alpha_1(q))/F_1(\alpha_1(m)) > 1.5$.

(c) To verify the quadrature formula (5.3) it is sufficient to consider the functions

$$F(z,\phi) = z^\lambda \cos\nu\phi, \qquad z^\lambda \sin\nu\phi, \qquad 0 \leq \lambda, \nu \leq p.$$

For $\nu > 0$ both the integral and the sum in (5.3) are equal to zero, cf. Theorem 3.1. In the remaining case $F(z,\phi) = z^\lambda$, formula (5.3) follows from the hypothesis of part (ii) and Bernstein's Theorem 2.1. That result and the inequality $m \le (p+2)/2$ also imply the upper bound for $N(p)$ in our theorem.

6. The Unit Disc

Let $D = D(0,1)$ be the closed unit disc in $\mathbf{R}^2$ described by $x = r\cos\phi$, $y = r\sin\phi$, $0 \le r \le 1$, $-\pi < \phi \le \pi$, or $0 \le \phi < 2\pi$. Systems of nodes in D will be denoted by

$$(x_j, y_j) \simeq (r_j, \phi_j), \qquad j = 1, \dots, N. \tag{6.1}$$

THEOREM 6.1. (i) *Suppose that the nodes* (6.1) *give a Chebyshev-type quadrature formula*

$$\frac{1}{\pi}\int_D f(x,y)\,dx\,dy \approx \frac{1}{N}\sum_{j=1}^{N} f(x_j, y_j) \tag{6.2}$$

of degree (*at least*) p. *Then*

$$N > \frac{1}{16\pi^2}p^3 > .0063p^3.$$

(ii) *For* $p \ge 2$, *let* m *be the smallest integer* $\ge (p+2)/4$ *and let* t_i, μ_i, *and* $N_1(2m-1)$ *provide a quadrature formula as in part* (ii) *of Bernstein's Theorem* 2.1. *Set*

$$r_i = \sqrt{(t_i+1)/2}, \qquad \phi_k = \frac{(2k-1)\pi}{p+1}, \qquad \phi_{i,l} = \frac{(2l-1)\pi}{\mu_i(p+1)},$$

$$N(p) = (p+1)N_1(2m-1).$$

Then for all polynomials $f(x,y) \simeq F(r,\phi)$ *of degree* $\le p$,

$$\begin{aligned}\frac{1}{\pi}\int_D f(x,y)\,dx\,dy &= \frac{1}{\pi}\int_0^1\int_0^{2\pi} F(r,\phi)\,r\,dr\,d\phi \\ &= \frac{1}{N(p)}\sum_{i=1}^{2m-1}\mu_i\sum_{k=1}^{p+1}F(r_i,\phi_k) \\ &= \frac{1}{N(p)}\sum_{i=1}^{2m-1}\sum_{l=1}^{\mu_i(p+1)}F(r_i,\phi_{i,l}).\end{aligned} \tag{6.3}$$

Here

$$N(p) < (p+1)\left\{\tfrac{1}{4}\sqrt{2}(p+9)(p+21)+2\right\} \le Cp^3,$$

with $C = .36$ *for large* p.

Proof. (i) Suppose that formula (6.2) with the nodes (6.1) has degree $p \ge 1$ and set $m = [(p+1)/2]$. Then for any polynomial $g(x)$ of degree $\le 2m-1$, taking $f(x,y) = g(x)$ in (6.2),

$$\frac{1}{\pi}\int_D f(x,y)\,dx\,dy = \frac{2}{\pi}\int_{-1}^{1} g(x)\sqrt{1-x^2}\,dx = \frac{1}{N}\sum_{j=1}^{N} g(x_j).$$

Thus by Theorem 2.2 with $\nu = 1$, using formula (2.10),

$$N \ge \frac{1}{\lambda_1^{(1)}(m)} = \frac{m+1}{2\sin^2(\pi/(m+1))} > \frac{(m+1)^3}{2\pi^2} > \frac{1}{16\pi^2}p^3.$$

(ii) It is sufficient to verify (6.3) for all monomials $x^\alpha y^\beta$ of degree $\lambda \le p$. Such monomials can be written as linear combinations of terms

$$F(r,\phi) = r^\lambda \cos(\lambda - 2\mu)\phi, \qquad F(r,\phi) = r^\lambda \sin(\lambda - 2\mu)\phi$$

with $0 \le 2\mu \le \lambda$. We will check (6.3) for the latter functions.

We first take $\lambda - 2\mu = q > 0$. In that case the integrals over D are equal to zero and also the sums: Since $0 < q \le p$, Theorem 3.1 shows that

$$\sum_{k=1}^{p+1} \cos q\phi_k = \sum_{k=1}^{p+1} \sin q\phi_k = 0,$$

$$\sum_{l=1}^{\mu_i(p+1)} \cos q\phi_{i,l} = \sum_{l=1}^{\mu_i(p+1)} \sin q\phi_{i,l} = 0.$$

It remains to consider the case $\lambda = 2\mu$ with $\mu \le p/2 \le 2m-1$. However, for $F(r,\phi) = r^{2\mu}$, formula (6.3) reduces to a special case of Bernstein's formula (2.4) when we substitute $r^2 = (t+1)/2$.

The upper bound for $N(p)$ follows from inequality (2.5) for $N_1(2m-1)$, combined with the observation that $m \le (p+5)/4$.

Remark 6.2. The solid cylinder $x^2+y^2 \le 1$, $-1 \le z \le 1$ in $\mathbf{R}^3$ may be treated as the product of $D(0,1)$ and the interval $[-1,1]$. Here a Chebyshev-type quadrature formula of degree p will require order p^5 nodes.

7. The Unit Sphere

Let $S = S(0,1)$ be the unit sphere in $\mathbf{R}^3$ described by

$$z = \cos\theta, \qquad x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi,$$
$$0 \le \theta \le \pi, 0 \le \phi < 2\pi,$$

with area element

$$d\sigma(x,y,z) = \sin\theta\, d\theta\, d\phi = |dz|d\phi.$$

Systems of nodes on S will be denoted by

$$\xi_j = (x_j, y_j, z_j) \simeq (\theta_j, \phi_j) \simeq (z_j, \phi_j), \qquad j = 1, \dots, N. \tag{7.1}$$

Theorem 7.1. (i) *Suppose that the nodes* (7.1) *give a Chebyshev-type quadrature formula*

$$\frac{1}{4\pi}\int_S f(x,y,z)\, d\sigma(x,y,z) \approx \frac{1}{N}\sum_{j=1}^{N} f(\xi_j) \tag{7.2}$$

of degree (*at least*) p, *that is*, (7.2) *is exact for all polynomials* $f(x,y,z)$ *of degree* $\le p$. *Then the number* N' *of distinct nodes must exceed* $p^2/4$, *hence*

$$N \ge N' > p^2/4.$$

(ii) *For* $p \ge 0$, *let* m *be the smallest integer* $\ge (p+1)/2$ *and let* t_i, μ_i *and* $N_1(2m-1)$ *provide a quadrature formula as in part* (ii) *of Bernstein's Theorem* 2.1. *Set*

$$z_i = t_i, \qquad \phi_k = \frac{(2k-1)\pi}{p+1}, \qquad \phi_{i,l} = \frac{(2l-1)\pi}{\mu_i(p+1)},$$
$$N(p) = (p+1)N_1(2m-1).$$

Then for all polynomials $f(x,y,z) \simeq F(z,\phi)$ *of degree* $\le p$,

$$\begin{aligned}
&\frac{1}{4\pi}\int_S f(x,y,z)\, d\sigma(x,y,z)\\
&\quad = \frac{1}{4\pi}\int_{-1}^{1}\int_0^{2\pi} F(z,\phi)\, dz\, d\phi\\
&\quad = \frac{1}{N(p)}\sum_{i=1}^{2m-1}\mu_i\sum_{k=1}^{p+1} F(z_i,\phi_k) = \frac{1}{N(p)}\sum_{i=1}^{2m-1}\sum_{l=1}^{\mu_i(p+1)} F(z_i,\phi_{i,l}).
\end{aligned} \tag{7.3}$$

Here

$$N(p) < \sqrt{2}\,(p+1)\{(p+4)(p+10)+\sqrt{2}\}.$$

For the proof we recall some simple facts about spherical harmonics, cf. [21, 24]. A spherical harmonic $Y_n(\xi)$ of order n is the restriction to S of a homogeneous harmonic polynomial $h_n(x, y, z)$ of degree n,

$$h_n(x,y,z) = r^n Y_n(\xi), \qquad (x,y,z) = r\xi, \qquad r \geq 0,\ \xi \in S.$$

The spherical harmonics of order n form a rotation invariant linear subspace H_n of $L^2(S)$ of dimension $2n+1$. A standard orthogonal basis of H_n is given by the functions

$$\begin{aligned} Y_{n,s}(z,\phi) &= (\sin\theta)^{|s|}\left(D^{|s|}P_n\right)(\cos\theta)e^{is\phi} \\ &= \left(1-z^2\right)^{|s|/2}P_n^{(|s|)}(z)e^{is\phi}, \qquad -n \leq s \leq n. \end{aligned} \tag{7.4}$$

The restriction to S of any polynomial $f(x, y, z)$ of degree q is equal to an element of the direct sum

$$V_q = H_0 \oplus H_1 \oplus \cdots \oplus H_q; \qquad \dim V_q = (q+1)^2. \tag{7.5}$$

Proof of Theorem 7.1. (i) Let the number of distinct nodes in the system (7.1) be N'. Define $q = [\sqrt{N'}]$ so that $(q+1)^2 > N'$. There will then be a nonzero linear combination G of basis elements of V_q which vanishes at each of the nodes $\xi_1, \ldots, \xi_{N'}$. Indeed, a system of N' homogeneous linear equations in more than N' unknowns always has a nonzero solution. Denoting the harmonic polynomial corresponding to G by g and setting $g\bar{g} = f_0$, we have

$$\deg f_0 \leq 2q, \qquad \int_S f_0\, d\sigma > 0, \qquad \sum_{j=1}^{N} f_0(\xi_j) = 0.$$

Thus formula (7.2) is not exact for f_0. Conclusion: if quadrature formula (7.2) with the nodes (7.1) has degree p, then

$$p < 2q \leq 2\sqrt{N'} \leq 2\sqrt{N}, \qquad N \geq N' > p^2/4.$$

(ii) Since every polynomial $f(x, y, z)$ of degree $\leq p$ reduces to an element $F(\theta, \phi) \simeq F(z, \phi)$ of the linear space V_p on S, it will be sufficient to verify the formulas (7.3) for the basis elements

$$F(z,\phi) = Y_{n,s}(z,\phi), \qquad n = 0, 1, \ldots, p,\ -n \leq s \leq n. \tag{7.6}$$

Now for $0 < |s| \leq p$, both the integral and the sums in (7.3) will vanish for this F:

$$\int_0^{2\pi} e^{is\phi}\, d\phi = 0, \qquad \sum_{k=1}^{p+1} e^{is\phi_k} = \sum_{l=1}^{\mu_j(p+1)} e^{is\phi_{j,l}} = 0,$$

cf. Theorem 3.1. It remains to consider the case $s = 0$,

$$F(z,\phi) = Y_{n,0}(z,\phi) = P_n(z), \qquad 0 \le n \le p.$$

However, in that case formula (7.3) reduces to a special case of formula (2.4) in Bernstein's Theorem 2.1.

The inequality for $N(p)$ follows from inequality (2.5) and the observation that $m \le (p+2)/2$.

COROLLARY 7.2. *The last member in* (7.3) *provides a Chebyshev-type quadrature formula of degree p for S with $N(p)$ distinct nodes, hence there exist so-called spherical t-designs in $\mathbf{R}^3$ consisting of $\mathscr{O}(t^3)$ points.*

We conjecture that there exist such designs consisting of $\mathscr{O}(t^2)$ points.

COROLLARY 7.3. *In the terminology of Section* 1,

$$p^2/4 < N_S(p) < \sqrt{2}(p+1)\{(p+4)(p+10) + \sqrt{2}\}.$$

Remarks 7.4. Lower bounds for $N_S(p)$ of order p^2 may also be obtained by using projections. Thus it can be shown that in a quadrature formula (7.2) of degree p, there must be a node ξ_j on every spherical cap of (spherical) radius $2 \arcsin j_1/p$, where j_1 is the first positive zero of the Bessel function J_0.

Treating the unit ball $B = B(0,1)$ in $\mathbf{R}^3$ as a kind of product of sphere and interval, we would obtain inequalities

$$c_1 p^4 \le N_B(p) \le c_2 p^5,$$

cf. Theorem 2.2 with $\nu = 3/2$.

The present methods readily extend to higher dimensions. For the d-dimensional unit sphere S^d in $\mathbf{R}^{d+1}$ our result would be

$$c_1 p^d \le N_{S^d}(p) \le c_2 p^{d(d+1)/2} \qquad (c_1 > 0).$$

Here we also expect that the lower bound gives the true order.

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